Assignment 7 Solution

Ex1:

$$f(1) = \frac{1}{2} = \frac{2^1 - 1}{2^1}$$

$$f(2) = \frac{3}{4} = \frac{2^2 - 1}{2^2}$$

$$f(3) = \frac{7}{8} = \frac{2^3 - 1}{2^3}$$

Conjecture:

$$f(n) = \frac{2^n - 1}{2^n}$$

Proof:

Assume true for n, prove it is true for n+1

$$f(n+1) = f(n) + \frac{1}{2^{n+1}} \left[? = \frac{2^{n+1} - 1}{2^{n+1}} \right]$$

$$f(n+1) = \frac{2^n - 1}{2^n} + \frac{1}{2^{n+1}}$$
$$= \frac{2 * (2^n - 1) + 1}{2^{n+1}}$$

$$= \frac{2^{n+1} - 1}{2^{n+1}}$$

Proved!

Ex2:

RTP: 6 divides $n^3 - n$ whenever n is a non-negative integer

- try for n = 0, and n =1, $n^3 - n$ = 0, which can be divided by 6 [0= 6k, for **k** belongs to **Z**, k =0]
- try for n = 2, and n =1, $n^3 - n = 8 - 2 = 6$, which can be divided by 6 also [6= 6k, for **k** belongs to **Z**, k =1]

$n^3 - n = n\,(n^2 - 1) = (n - 1) * n * (n + 1)$

Now,

- if n is even, then n = 2k for some **k** belongs to **Z**
- if n is odd, then n=2k+1, then n-1 = 2k, for some **k** belongs to **Z**
- **==>, whether n is odd or even,** $(n - 1) * n * (n + 1)$ **is divisible by 2, i.e :** $(n - 1) * n * (n + 1) = 2j$ for some **j** belongs to **Z**

- if n is divisible by 3, then n = 3m for some **m** belongs to **Z**
- else if n is not divisible by 3, then n=3m+1, or n=3m+2 for some **m** belongs to **Z**
  - if n = 3m+1, then n-1 = 3m, then n-1 is divisible by 3
  - if n = 3m+2, then n+1 = 3m+3 = 3(m+1), then n+1 is divisible by 3
- **==> whether n is divisible by 3 or not,** $(n-1)*n*(n+1)$ **is divisible by 3, i.e :** $(n-1)*n*(n+1) = 3i$ for some **i** belongs to **Z**
- **then** $(n-1)*n*(n+1)$is always divisible by 2 and 3, then
- $n^3 - n = (n-1)*n*(n+1) = 2*3*p = 6p$, for some **p** belongs to **Z**

Then $n^3 - n$ **is always divisible by 6**

Ex3:

The problem is that it wasn't verified on a base case.
For n = 1, $\sum_{i=1}^{1} i = 1$,

but according to the given formula, $\sum_{i=1}^{1} i = \frac{\left(1+\frac{1}{2}\right)^2}{2} = 1.125$, so it fails on base cases.

Ex4:

Base Case: n = 1, m > 1, f : {1,2…m} → {1}, then there exists more than one value in the domain that maps to 1, then f is not 1:1 by definition

Inductive Step:
Assume for all n < m, f : {1,2…m} → {1, 2, … n}, then f is not 1:1
Show that it's still holds for n+1 < m, i.e: f : {1,2…m} → {1, 2, … n+1} is also not 1:1

Case 1*:* if no value in domain maps to n+1, then the range can be shrunk to {1, 2…n}, **then f is not 1:1 from inductive step**

Case 2*:* if more than 1 value in domain maps to n+1, **then f is not 1:1 by def**

Case 3: exactly 1 value (k) from domain maps to n+1, then we can remove k from domain and n+1 from range *[and replace k by m, without affecting the function]*, to have f: {1,2 … m-1} → {1, 2…n}, i.e: f: {1,2 … m'} → {1, 2…n}, and m-1 > n since m > n+1, **then f is not 1:1 by inductive hypothesis**

Then in all cases f is not 1:1 when n < m

Ex5:
a) By using the 4-cent stamps, we can have 4, 8, 12, 16, 20, 24…

Starting with a 7, and by adding 4, we can have 7, 11, 15, 19, 23…
Starting with 2 7s, we can have 14, 18, 22…

By starting with 3 7s, we can have 21, 25, 29…
Since we had 4 numbers in a row, we can know the numbers that can be formed. We have 18, 19, 20 and 21

- ⇨ For n ≥ 18, we can right any number by using only 4s and 7s and to prove that we use induction and strong induction in part b) and c)
- ⇨ The numbers that can be formed are 4, 7, 8, 11, 12, 14, 15, 16 and **any n ≥ 18**

b) <u>RTP:</u> For all k >= 18, k = 4a + 8b for a,b >= 0

18 = 2(7) + 4 [basis step]

Assume it's true for k, prove its true for k+1

<u>Case 1:</u> if k has one of more 7s (b>0), then b-1 > 0,
4(a+2) + 7(b-1) = 4a + 7b +1 = k+1

<u>Case 2:</u> if k has no 7s (b=0, k = 4a), then a >= 5, since k>=18, and 18 = 4a, then a>= 4.5, but a is integer, then a >= 5
Then a-5 >= 0, then 4(a-5) + 7(3) = 4a + 1 = k+1

Then in both cases we can move from build k+1 from stamps of 4 and 7 if k>=18, by induction

c) <u>RTP:</u> we can right any number above 18 by only using 4s and 7s

Let P(n) be the proposition that postage of n cents can be formed using 4-cent and 7-cent stamps

- P(18) uses 2 7s and 1 4 [basis]
- P(19) uses 1 7 and 3 4s [basis]
- P(20) uses 5 4s [basis]
- P(21) uses 3 7s [basis]

Suppose P(j) holds for 18 ≤ j ≤ k where k ≥ 21 [inductive step] [hypothesis]
RTP: P(k+1) holds
k ≥ 21 => k – 3 ≥ 18

- ⇨ We can form k – 3 by only using 7s and 4s
- ⇨ By adding only 4, we form k + 1 by only using 7s and 4s

In the mathematical induction, we supposed for a particular k that the hypothesis works, while in the strong induction we supposed that for all the values between 18 and k the hypothesis works

<u>Ex6:</u>

<u>Base Case:</u> $1 = 2^0$, $3 = 2^0 + 2^1$

<u>RTP:</u> Any integer can be written as sum of distinct powers of 2

<u>Inductive step:</u>
Assume that $for\ all\ k \leq n, k = 2^{b_0} + 2^{b_1} + \cdots + 2^{b_{m-1}} + 2^{b_m},$
$such\ that\ all\ b_i s\ are\ distinct\ values$

Show that k+1 can also be written as a sum of distinct powers of 2

<u>Case 1:</u> if k is even, then there is no $b_i=0$, since 2^x is even for all x > 0, and sum of even numbers is even, so k can have 2^0 in its sum… and k+2^0 = k+1, **then k+1 can be written as sum of distinct powers of 2**

<u>Case 2:</u> if k is odd, then k+1 is even, then (k+1)/2 can be written as sum of distinct powers of 2 (By Strong Induction Hypothesis),
then $\frac{k+1}{2} = 2^{b_0} + 2^{b_1} + \cdots + 2^{b_{m-1}} + 2^{b_m}$ where the powers of 2 are distinct
then $(k + 1) = 2 * (2^{b_0} + 2^{b_1} + \cdots + 2^{b_{m-1}} + 2^{b_m}) = 2^{1+b_0} + 2^{1+b_1} + \cdots + 2^{1+b_{m-1}} + 2^{1+b_m}$, then the powers of 2 are still distinct **then k+1 can be written as sum of distinct powers of 2**

in the two cases, k+1 can be written as sum of distinct powers of 2

<u>Ex7:</u>

<u>RTP:</u> $2n^2 - 10n + 4 \geq 0\ whenever\ n \geq 5$

<u>Base case:</u> n =5, $2n^2 - 10n + 4 = 4 \geq 0, so\ true$

<u>Assume it holds for n, prove it true for n+1</u>
$f(n + 1) = 2(n + 1)^2 - 10(n + 1) + 4$
$= 2(n^2 + 2n + 1) - 10n - 10 + 4$
$= 2n^2 + 4n + 2 - 10n - 10 + 4$
$= f(n) + 4n - 8$

now, is $f(n) + 4n - 8 \geq 0\ for\ n \geq 5$?
$n \geq 5\ then\ 4n \geq 20\ then\ 4n - 8 \geq 12 \geq 0$
$\boldsymbol{So}: f(n) \geq 0\ and\ 4n - 8 \geq 0\ for\ n \geq 5, then\ f(n + 1) \geq 0\ for\ n \geq 5$

then n =5, $2n^2 - 10n + 4 \geq 0, for\ n \geq 5\ by\ induction$

<u>Ex8:</u>

<u>RTP:</u> $(A_1 \cap A_2 \cap \cdots \cap A_n) \cup B = (A1 \cup B) \cap (A2 \cup B) \cap \cdots \cap (A_n \cup B)$

<u>Base Case:</u> For n=1, $A_1 \cup B = A_1 \cup B$

$P(k) = (A_1 \cap A_2 \cap \cdots \cap A_k) \cup B$,
$Q(k) = (A1 \cup B) \cap (A2 \cup B) \cap \cdots \cap (A_k \cup B)$,

<u>Inductive Step:</u>
Assume P(k) = Q(k), show P(k+1) = Q(k+1)

Q(k+1) = $(A1 \cup B) \cap (A2 \cup B) \cap \cdots \cap (A_k \cup B) \cap (A_{k+1} \cup B)$
$= Q(k) \cap (A_{k+1} \cup B)$
$= P(k) \cap (A_{k+1} \cup B)$
$= P(k) \cap (A_{k+1} \cup B)$
$= ((A_1 \cap A_2 \cap \cdots \cap A_k) \cup B) \cap (A_{k+1} \cup B)$
$= ((A1 \cup B) \cap (A2 \cup B) \cap \cdots \cap (A_k \cup B)) \cap (A_{k+1} \cup B)$
$= (A1 \cup B) \cap (A2 \cup B) \cap \cdots \cap (A_k \cup B) \cap (A_{k+1} \cup B)$

Proved

<u>Ex9:</u>
$n^3 + (n+1)^3 + (n+2)^3 = 3(n^3 + 3n^2 + 5n + 3)$
$(n+1)^3 + (n+2)^3 + (n+3)^3 = 3(n^3 + 6n^2 + 14n + 12)$